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Title: FERMIONIC LINEAR OPTICS AND MATCHGATES

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Fermionic Linear Optics and Matchgates

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Abstract

Fermionic linear optics is efficiently classically simultable. Here it is shown that the set of states achievable with fermionic linear optics and particle measurements is the closure of a low dimensional Lie group. The weakness of fermionic linear optics and measurements can therefore be explained and contrasted with the strength of bosonic linear optics with particle measurements. An analysis of fermionic linear optics is used to show that the two-qubit matchgates and the simulatable matchcircuits introduced by Valiant generate a monoid of extended fermionic linear optics operators.

1 Introduction

It is conjectured that standard quantum computation is more efficient than probabilistic computation. The conjecture is supported by the ability to efficiently factor large numbers [1] and simulate physics [2] using quantum computers, by proofs that quantum computers are more powerful with respect to some black boxes [3], and by results [4] showing exponential improvements in communication complexity.

To delineate the conjecture one can consider models of computation where the basic operations are multiplication of linear operations in a given set G. Each operation in G is associated with a complexity (e.g. the length of its name), so that the complexity of a product $g_1 g_2 \ldots$ is the sum of the complexities of the g_i. One can then ask questions about the complexity of calculating qantities like: 1. Computing the entries in a standard basis of a product. 2. Computing the trace of a product. When G is the a set of elementary quantum gates, the power of quantum computers is equivalent to being able to efficiently sample from a probability distribution with expectation an entry of a product and variance $O(1)$ (see [5]). The power of one-bit quantum computers [5] is equivalent to sampling from a probability distribution with expectation the trace of a product and variance $O(2^n)$, where n is the number of qubits.

A special case is when the set G is the group of operators normalizing the group generated by the Pauli matrices (bit flip, sign flip). For n qubits, this group has order $2^{O(n^2)}$ and plays a crucial role in encoding and decoding stabilizer codes [6] and in fault tolerant quantum computation [7]. In [8] it is shown that even when this group is extended by projections onto the logical states of qubits, the complexities of the two questions above are polynomial. Two similarly defined groups are the linear optics operators for fermions and bosons. In both cases, the groups are Lie groups of polynomial dimension in the number of modes. (Modes play the same role as qubits in these systems). A few simulatability results were known for these groups. For example, for bosons, the orbit of the vacuum state under the linear operations consists of Gaussian states, for which many relevant quantities can be efficiently computed. Similarly, particle preserving linear operations applied to exactly one boson lead only to states that are equivalent to classical waves [9].

Recently, Valiant [10] demonstrated a set of products of operators (those definable by a class of "matchcircuits") for which the complexities of the first questions and many of its generalizations are polynomial. Terhal and DiVincenzo [11] realized that this set includes the unitary linear operations for fermions and that as a consequence, it is unlikely that it is possible to realize quantum computation in fermions by means of linear operations and particle detectors with feedback. They give a direct and efficient simulation of these operations based on fermionic principles. This result is at first surprising: In [12] it was shown that with bosons, linear operations and particle detectors with feedback are sufficient for realizing quantum computation. The difference between fermions and bosons is explained by realizing that the effects of particle detectors are expressible as limits of non-unitary linear operations in fermions but not in bosons. As a result, the states achievable with fermionic linear operations and particle measurements are in the closure of a "simple" set.

Since matchgate operations are non-unitary, one can ask what additional power is provided by Valiant's simulation of matchgates. Here it is shown that the two-qubit matchgates generate the monoid which is the closure of a group of extended fermionic linear operators in the Jordan-Wigner representation [13]. This group defines the nondeterministic computations that can be physically realized with unitary linear operation and particle measurements. The equivalence of two-qubit matchgates and linear fermionic two-qubit operations generalizes to the set of simulatable matchcircuits introduced by Valiant.

2 Linear Fermionic Operations

Let $I, X^{(k)}, Y^{(k)}, Z^{(k)}$ denote the identity and the Pauli operators acting on qubit k. Define $U_k = Z^{(1)} \ldots Z^{(k-1)} U^{(k)}$ ($U_1 = U^{(1)}$) for $U = X, Y$. Then the U_k define a representation of fermionic mode operators. In particular, $(X_k + iY_k)/2$ and $(X_k - iY_k)/2$ represent the annilation and creation operators for mode k. It is straightforward to check the appropriate commutation and anti-commutation relationships. Let $\mathcal{L}_1$ be the linear span of the identity together with the U_k for $1 \leq k \leq n$, where n is the number of qubits. The set $\mathcal{G}$ of *fermionic linear optics operators* is the set of invertible matrices that preserve $\mathcal{L}_1$ by conjugation. That is, $g \in \mathcal{G}_1$ iff for all $A \in \mathcal{L}_1$, $gAg^{-1} \in \mathcal{L}_1$. The terminology refers to the property that conjugation of an annihilation or a creation operator results in a linear combination of such operators. Let $\mathcal{L}_2$ be the set of products of two operators in $\mathcal{L}_1$, so that $\mathcal{L}_2 = \mathcal{L}_1\mathcal{L}_1$. The group $\mathcal{G}_2$ of *extended* linear optics operators is the set of invertible matrices that preserve $\mathcal{L}_2$. Note that $\mathcal{G}_1 \subseteq \mathcal{G}_2$. (In bosons, the analogous definitions lead to identical groups.) The group $\mathcal{G}_2$ is considered to be "unphysical" for fermions, due to the presence of odd products of annihilation and creation operators. Nevertheless, the present work shows that it is interesting and useful.

The space $\mathcal{L}_2$ is a (complex) Lie algebra. It is spanned by Pauli operator products given by I, $U^{(k)}$, $Z^{(k)}$, and $U^{(k)}Z^{(k+1)}\ldots Z^{(k+l)}V^{(k+l+1)}$ with $U, V \in \{X, Y\}$. The dimension of $\mathcal{L}_2$ is $2n^2+n+1$. By considering general sums of Pauli products, one can check that if for every $A \in \mathcal{L}_2$, $[X, A] \in \mathcal{L}_2$, then $X \in \mathcal{L}_2$. It follows that $\mathcal{L}_2$ is the Lie algebra of $\mathcal{G}_2$. All strictly quadratic (in $\mathcal{L}_1$) terms of $\mathcal{L}_2$, together with the identity also form a Lie algebra $\mathcal{L}_2'$ of dimension $2n^2 - n$, which is the Lie algebra of $\mathcal{G}_1$. Physically, realizable operators are continuously generated from the identity. As a result, for the remainder of the paper, $\mathcal{G}_i$ is assumed to be given by the exponentials of $\mathcal{L}_i$.[1]

In using (extended) linear operators for computation, one starts with the vacuum state $|\mathrm{v}_n\rangle = |0\ldots0\rangle_{1\ldots n}$ and applies operators in $\mathcal{G}_1$ ($\mathcal{G}_2$) and measurements in the number basis $|0\rangle, |1\rangle$. The outcomes of measurements are described by applying the measurement projections $|0\rangle\langle 0| = \frac{1}{2}(I + Z^{(k)})$ and $|1\rangle\langle 1| = \frac{1}{2}(I + Z^{(k)})$. For standard computation, which projection "happens" is determined by the square amplitude of the result of applying it. For nondeterministic computation we can "choose" the outcome. In either case, analysis of the capabilities requires studying products of operators in $\mathcal{G}_i$ and the measurement projections. Let $\mathcal{S}_i$ be the monoid given by closure of $\mathcal{G}_i$.

If $\mathcal{S}_2$ could be used for efficient faithful quantum computation, then $\mathcal{S}_2|\mathrm{v}\rangle$ has to contain sufficiently large subspaces. That is, the 2^m dimensional state space of m qubits must be contained in $\mathcal{S}_2|\mathrm{v}_n\rangle$ with $n = O(\mathrm{poly}(n))$. The following theorem makes this unlikely.

Theorem 1 *$\mathcal{S}_2$ is contained in the closure of $\mathcal{G}_2$.*

Proof. This is a consequence of the fact that the measurement projections are limits of elements of $\mathcal{G}_2$:

$$\frac{1}{2}(I + Z^{(k)}) \quad = \quad \lim_{t\to\infty} e^{tZ^{(k)}}/e^t$$

[1]Without a proof that this assumption holds, it is possible that the groups studied here are only the component of the identity of the originally defined groups.

$$\frac{1}{2}(I - Z^{(k)}) = \lim_{t\to\infty} e^{-tZ^{(k)}}/e^t \tag{1}$$

■

Since $\mathcal{G}_2$ is a $2n^2 + n + 1$-dimensional Lie group, Thm. 1 implies that $\mathcal{S}_2|\mathrm{v}_n\rangle$ is the closure of a small dimensional space. This suggests that $\mathcal{S}_2$ is not sufficiently strong for quantum computation. The fact that the normalizer of the Pauli group together with standard measurements are insufficient [8] follows in a similar way. That is, applying normalizer operators and projections onto stabilizer codes to the standard initial state results always in stabilizer states.

Note that a similar result can not be shown for bosonic linear operators with particle measurements. Only the projection operator onto the 0 boson state of a system is expressible as a limit of (non-unitary) linear operators. This explains why efficient linear optics quantum computation is possible [12].

3 Matchgates and Linear Operations

In [10], Valiant introduced a family of linear operators (called matchgates) acting on qubits using a graph theoretic construction, and showed that under certain conditions, the coefficients of matrices defined by products of matchgates could be efficiently calculated. Matchgates acting on two qubits were shown to satisfy a set of 5 equations, the matchgate identities. If B is the matrix defined by a matchgate acting on two qubits, then the following are 0:

$$M_1 = \langle 00|B|00\rangle\langle 11|B|11\rangle - \langle 10|B|10\rangle\langle 01|B|01\rangle - \langle 00|B|11\rangle\langle 11|B|00\rangle + \langle 10|B|01\rangle\langle 01|B|10\rangle \tag{2}$$

$$M_2 = \langle 10|B|00\rangle\langle 11|B|11\rangle - \langle 10|B|10\rangle\langle 11|B|01\rangle - \langle 11|B|00\rangle\langle 10|B|11\rangle + \langle 10|B|01\rangle\langle 11|B|10\rangle \tag{3}$$

$$M_3 = \langle 01|B|00\rangle\langle 11|B|11\rangle + \langle 01|B|01\rangle\langle 11|B|10\rangle - \langle 11|B|00\rangle\langle 01|B|11\rangle - \langle 01|B|10\rangle\langle 11|B|01\rangle \tag{4}$$

$$M_4 = \langle 00|B|01\rangle\langle 11|B|11\rangle + \langle 01|B|01\rangle\langle 10|B|11\rangle - \langle 00|B|11\rangle\langle 11|B|01\rangle - \langle 10|B|01\rangle\langle 01|B|11\rangle \tag{5}$$

$$M_5 = \langle 00|B|10\rangle\langle 11|B|11\rangle - \langle 10|B|10\rangle\langle 01|B|11\rangle - \langle 00|B|11\rangle\langle 11|B|10\rangle + \langle 01|B|10\rangle\langle 10|B|11\rangle \tag{6}$$

Let $\mathcal{M}_2$ be the set of matrices B satisfying the identities $M_i = 0$ and either $\langle 11|B|11\rangle \neq 0$ or B is diagonal. Valiant showed that these matrices are realizable by matchgates.

Theorem 2 *The closure of $\mathcal{M}_2$ is $\mathcal{S}_2$ for two qubits.*

Proof. The Lie algebra which generates $\mathcal{S}_2$ is spanned by the 11 operators

$$L = \{II, XI, YI, ZI, ZX, ZY, XX, XY, YX, YY, IZ\} \tag{7}$$

Here UV abbreviates $U^{(1)}V^{(2)}$. One can check that for $A \in L \setminus \{II\}$, $A(YX) + (YX)A^T = 0$: It suffices to note that if $A^T = A$, then A anticommutes with YX, and if $A^T = -A$, which is the case if A contains an odd number of Y's, then A commutes with YX. (This property generalizes for arbitrary number of qubits, using the operator $YXYX\ldots$ instead of YX.) The identity $A(YX) + (YX)A^T = 0$ can be re-written in the form $(A \otimes I + I \otimes A)T = 0$, where T is the antisymmetric vector

$$T = |00\rangle|11\rangle - |11\rangle|00\rangle + |01\rangle|10\rangle - |10\rangle|01\rangle. \tag{8}$$

This means that T is an eigenvector of the Lie group $\mathcal{L}$ generated by $L \oplus L = \{A \otimes I + I \otimes A : A \in L\}$. Note that $\mathcal{L} = \{B \otimes B : B \in \mathcal{G}_2\}$. $\mathcal{L}$ preserves antisymmetric vectors, so the statement that $\mathcal{L}T \propto T$ is equivalent to $R^T \mathcal{L} T = 0$ for all R antisymmetric such that $R^T T = 0$. The dimension of such R is 5, and here is a basis:

$$\begin{aligned} R_1 &= |00\rangle|11\rangle - |11\rangle|00\rangle - |01\rangle|10\rangle + |10\rangle|01\rangle && (9)\\ R_2 &= |00\rangle|01\rangle - |01\rangle|00\rangle && (10)\\ R_3 &= |00\rangle|10\rangle - |10\rangle|00\rangle && (11)\\ R_4 &= |01\rangle|11\rangle - |11\rangle|01\rangle && (12)\\ R_5 &= |10\rangle|11\rangle - |11\rangle|10\rangle && (13) \end{aligned}$$

Define the expressions

$$\begin{aligned} E_i &= R_i^T B T && (14)\\ E_i^T &= T^T B R_i^T && (15) \end{aligned}$$

Since for two qubits $\mathcal{L}_2^T = \mathcal{L}_2$, members B of $\mathcal{G}_2$ satisfy the identities $E_i = 0, E_i^T = 0$. Because these identites are all derived from an eigenvector condition, the set of matrices B satisfying them is a monoid $\mathcal{G}_2'$ containing $\mathcal{G}_2$.

Using the equivalence

$$(|ab\rangle|cd\rangle)^T B \otimes B(|ef\rangle|gh\rangle) = \langle ab|B|ef\rangle\langle cd|B|gh\rangle, \tag{16}$$

one can check that the following hold

$$E_1 + E_1^T = 4M_1 \tag{17}$$

$$
\begin{aligned}
E_4 &= 2M_3 && (18)\\
E_5 &= 2M_2 && (19)\\
E_4^T &= 2M_4 && (20)\\
E_5^T &= 2M_5 && (21)\\
\langle 11|B|11\rangle(E_1 - E_1^T) &= 4(\langle 01|B|11\rangle M_2 - \langle 10|B|11\rangle M_3 + \langle 11|B|10\rangle M_4 - \langle 11|B|01\rangle M_5) && (22)\\
\langle 11|B|11\rangle E_2 &= 2(\langle 01|B|11\rangle M_1 - \langle 00|B|11\rangle M_3 + \langle 01|B|10\rangle M_4 - \langle 01|B|01\rangle M_5) && (23)\\
\langle 11|B|11\rangle E_3 &= 2(\langle 10|B|11\rangle M_1 - \langle 00|B|11\rangle M_2 - \langle 10|B|01\rangle M_5 + \langle 10|B|10\rangle M_4) && (24)\\
\langle 11|B|11\rangle E_2^T &= 2(\langle 11|B|01\rangle M_1 - \langle 11|B|00\rangle M_4 + \langle 10|B|01\rangle M_3 - \langle 01|B|01\rangle M_2) && (25)\\
\langle 11|B|11\rangle E_3^T &= 2(\langle 11|B|10\rangle M_1 - \langle 11|B|00\rangle M_5 - \langle 01|B|10\rangle M_2 + \langle 10|B|10\rangle M_3) && (26)
\end{aligned}
$$

Mathematica instructions to check the above relationships are included verbatim in Appendix A.

Since diagonal matrices trivially satisfy $E_i = 0$, $E_i^T = 0$ $(i > 1)$ and $E_1 - E_1^T = 0$, the identites imply that $\mathcal{M}_2 \subseteq \mathcal{G}_2'$. Let $\mathcal{M}_2' = \{B \in \mathcal{M}_2 : \langle 11|B|11\rangle \neq 0\}$. By directly solving for the entries of B other than $\langle 11|B|11\rangle$ in the first summand of the M_i, one can see that $\mathcal{M}_2'$ is an analytically coordinatizable 11 complex dimensional manifold. The diagonal members of $\mathcal{M}_2$ are in the closure of $\mathcal{M}_2'$.

The identities also imply that the elements of $\mathcal{G}_2$, and therefore those of $\mathcal{S}_2$, satisfy $M_i = 0$. It follows that the $B \in \mathcal{S}_2$ with B diagonal or $\langle 11|B|11\rangle \neq 0$ are in $\mathcal{M}_2$.

For invertible B, the identites $E_i = 0$ imply that $B(XY)B^T = \lambda XY$ for $\lambda \neq 0$. It follows that the tangent space at B is exactly that of $\mathcal{G}_2$ at B. Consequently, $\mathcal{M}_2'$ and $\mathcal{G}_2$ contain the same invertible matrices satisfying $\langle 11|B|11\rangle \neq 0$. It remains to show that these matrices are dense in both sets. For $\mathcal{M}_2'$ it suffices to observe that for fixed $\langle 11|B|11\rangle \neq 0$, their is an invertible $B \in \mathcal{M}_2'$, which implies that the determininant function is not null on this linearly defined subset. Hence the complement of the determinant's null set is dense. For $\mathcal{G}_2$ the density property follows from the fact that the subgroup generated by XI and XX acts transitively on the basis states.

■

4 Simulatable Matchcircuits

Valiant showed that any composition of operators consisting of two qubit matchgates on the first two qubits and gates of the form $e^{t(X^{(k)}X^{(k+1)})}$ and $e^{t(Y^{(k)}Y^{(k+1)})}$ is efficiently simulatable in the following sense: If B is a product of m such gates, then many sums of squares or square norms of entries of B can be computed efficiently in m and n (the number of qubits). Let $\mathcal{M}$ be set of all products of the gates mentioned.

Theorem 3 *The closure of $\mathcal{M}$ is $\mathcal{S}_2$.*

Proof. By definition and by Thm. 2, $\overline{\mathcal{M}} \subseteq \mathcal{S}_2$. It suffices to show that the invertible operators in $\mathcal{M}$ generate $\mathcal{G}_2$. This can be checked directly by using the Bloch sphere rules for conjugating products of Pauli matrices by 90° rotations ($e^{-iU\pi/4}$) around other products [14]. For example, $Z^{(1)}Z^{(2)}X^{(3)}$ is obtained by conjugating $Z^{(1)}Y^{(2)}$ with a rotation around $X^{(2)}X^{(3)}$. The operator $Z^{(3)}$ is obtained by conjugating $Z^{(1)}Z^{(2)}X^{(3)}$ with a rotation around $Z^{(1)}Z^{(3)}Y^{(3)}$. The latter operator can be deduced similarly to the way $Z^{(1)}Z^{(2)}X^{(3)}$ was obtained. Induction can be used to extend to arbitrarily many qubits. ■

5 Concluding Comments

It is true that bosons can be represented by paired fermions. So why does this not lead to an efficient realization of quantum computers by using this representation together with techniques for bosonic linear optics? One answer is that the bosonic linear operators in this representation correspond to Hamiltonians that are quartic in the annihilation and creation operators and are therefore not in $\mathcal{L}_2$. It is in fact not hard to see that adding to $\mathcal{L}_2$ only the Hamiltonian $Z^{(1)}Z^{(2)}$, the Lie algebra generated contains all products of Pauli matrices and so generates all invertible matrices [15].

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A Checking the Matchgate Identites

```
(* Mathematica notes. *)

(* Useful rules: *)
Unprotect[Dot];
Dot[tensor[a_,b_],tensor[c_,d_]] = (a.c)*(b.d);
Dot[-a_,b_] = -(a.b);
Dot[a_,-b_] = -(a.b);
Dot[-a_,-b_] = (a.b);

(* For obtaining the equation for the transpose: *)
trnsprls = {b[c_].k[d_] -> b[d].k[c]};
(* For obtaining the equation for the conjugate by XX: *)
xxrls = {x00->x11,x01->x10,x10->x01, x11->x00};
(* Swapping: *)
swprls = {x01->x10,x10->x01};
lswprls = {b[x01]->b[x10],b[x10]->b[x01]};

(* Conventions:
*  b[xab] stands for $\bra{ab}$, k[xab] for $\ket{ab}$.
*  Quadradic expressions for a matrix B are expressed
*  $\trace(X (B\tensor B))$ with X in the appropriate
*  tensor product space. X is given for various expressions.
*  This way the expression (b[x00].k[x00])*(b[x11].k[x01])
*  refers to the product $\bra{00}B\ket{00}\bra{11}B\ket{01}$.
*)

(* Matchgate expressions: *)
M1 = tensor[b[x00],b[x11]].tensor[k[x00],k[x11]] +
     - tensor[b[x10],b[x01]].tensor[k[x10],k[x01]] +
     - tensor[b[x00],b[x11]].tensor[k[x11],k[x00]] +
     + tensor[b[x10],b[x01]].tensor[k[x01],k[x10]];
M2 = tensor[b[x10],b[x11]].tensor[k[x00],k[x11]] +
     - tensor[b[x10],b[x11]].tensor[k[x10],k[x01]] +
     - tensor[b[x11],b[x10]].tensor[k[x00],k[x11]] +
```

```
     + tensor[b[x10],b[x11]].tensor[k[x01],k[x10]];
M3 = tensor[b[x01],b[x11]].tensor[k[x00],k[x11]] +
     + tensor[b[x01],b[x11]].tensor[k[x01],k[x10]] +
     - tensor[b[x11],b[x01]].tensor[k[x00],k[x11]] +
     - tensor[b[x01],b[x11]].tensor[k[x10],k[x01]];
M4 = tensor[b[x00],b[x11]].tensor[k[x01],k[x11]] +
     + tensor[b[x01],b[x10]].tensor[k[x01],k[x11]] +
     - tensor[b[x00],b[x11]].tensor[k[x11],k[x01]] +
     - tensor[b[x10],b[x01]].tensor[k[x01],k[x11]];
M5 = tensor[b[x00],b[x11]].tensor[k[x10],k[x11]] +
     - tensor[b[x10],b[x01]].tensor[k[x10],k[x11]] +
     - tensor[b[x00],b[x11]].tensor[k[x11],k[x10]] +
     + tensor[b[x01],b[x10]].tensor[k[x10],k[x11]];

(* Check:
M3 - (M4/.trnsprls)
 * = 0 *
 *
M2 - (M5/.trnsprls)
 * = 0 *
*)

(* Lie expressions: *)
T = tensor[k[x00],k[x11]] - tensor[k[x11],k[x00]] +
    tensor[k[x01],k[x10]] - tensor[k[x10],k[x01]];
R1 = tensor[b[x00],b[x11]] - tensor[b[x11],b[x00]] +
     tensor[b[x10],b[x01]] - tensor[b[x01],b[x10]];
R2 = tensor[b[x00],b[x01]] - tensor[b[x01],b[x00]];
R3 = tensor[b[x00],b[x10]] - tensor[b[x10],b[x00]];
R4 = tensor[b[x01],b[x11]] - tensor[b[x11],b[x01]];
R5 = tensor[b[x10],b[x11]] - tensor[b[x11],b[x10]];

E1 = Distribute[R1.T];
ET1 = E1/.trnsprls;
E2 = Distribute[R2.T];
```

```
ET2 = E2/.trnsprls;
E3 = Distribute[R3.T];
ET3 = E3/.trnsprls;
E4 = Distribute[R4.T];
ET4 = E4/.trnsprls;
E5 = Distribute[R5.T];
ET5 = E5/.trnsprls;
(* Check:
Simplify[E1+ET1  - 4*M1]
 * = 0 *
 *
Simplify[E4 - 2*M3]
 * = 0 *
 *
Simplify[E5 - 2*M2]
 * = 0 *
 *
Simplify[ (b[x11].k[x11])* E2 -
  2* (
     (b[x01].k[x11])*M1 +
    -(b[x00].k[x11])*M3 +
     (b[x01].k[x10])*M4 +
    -(b[x01].k[x01])*M5
  ) ]
 * = 0 *
 *
Simplify[ (b[x11].k[x11])* E3 -
  2* (
     -(b[x00].k[x11])*M2 +
     -(b[x10].k[x01])*M5 +
      (b[x10].k[x10])*M4 +
      (b[x10].k[x11])*M1
  ) ]
 * = 0 *
 *
```

```
Simplify[(M1/.trnsprls) - M1]
 * = 0 *
 *
Simplify[(M2/.lswprls) - M3]
 * = 0*
 *
Simplify[(E2/.lswprls)-E3]
 * = 0*
 *
Simplify[(b[x11].k[x11])*(E1 - (E1/.trnsprls)) -
  4* (
     b[x01].k[x11]*M2 +
    -b[x10].k[x11]*M3 +
    -b[x11].k[x01]*M5 +
     b[x11].k[x10]*M4
  ) ]
 * = 0*
 *
 * This confirms the identites claimed in the text.
*)
```